PREDICTING THE INTERVALS OF OSCILLATION FOR THE GEGENBAUER DIFFERENTIAL EQUATION

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ABSTRACT

In this paper we address the prediction of intervals of oscillation for the *Gegenbauer differential equation*, an ordinary linear homogeneous differential equation of the second order of the form:

$$(1-x^2)y'' - (2\mu+1)xy' + n(n+2\mu)y = 0,$$

for $|x| < 1$, where μ is a random constant, and $n > 0$. We present a method for determing the intervals of oscillation in solutions of second order linear homogeneous differential equation in general and subsequently, we apply it to our equation. The primary approach involves transforming the second-order linear homogeneous differential equation into one with constant coefficient. This transformation is achieved by through several changes in the independent variable x bringing the equation to its simplest canonical form. These transformations, we identify the intervals of oscillatory solutions using *Sturm's third comparison theorem*, which compares the coefficients of our equation with those of a simple harmonic oscillator equation. For fixed values of μ, we derive well-known equations from the *Gegenbauer differential equation*, such as *Chebyshev's equation* of first and second types, *Legendre's equation*, etc. Therefore, this method can also be applied to determine the intervals of oscillatory solutions for other second-order *linear* differential *equations in mathematical physics.*
Keywords: second order linear homogeneous differential equation, Interval of oscillation, *Gegenbauer differential equation*, *Sturm* comparison theorem, transformation of variables

1. INTRODUCTION

Differential equations find applications in numerous branches of science and technology, demanding accurate and efficient solution. While only few problems allow for analytic solutions, resolving a broad class of problems typically requires either numerical solutions or qualitative analyses without explicitly finding their solutions. The most effective analytical methods for solving differential equations often involve transforming them into simpler forms. In our previous works (Koçi 2015; Hoxha 2023), we investigated the following differential equation with variable coefficients:

$$L(y) = y^{(n)} + p_{n-1}(x)y^{n-1} + \cdots + p_1(x)y' + p_0(x)y = f(x), \qquad (E)$$

where, $p_0(x), p_1(x), \dots, p_{n-1}(x)$ are continuous functions on a given open interval (a, b). This equation remains linear but can be transformed into one with constant coefficients using the substitution

$$t = \varphi(x) = \frac{1}{\sqrt[n]{a}} \int [p_0(x)]^{\frac{1}{n}} dx.$$

Transforming an equation with variable coefficients into one with constant coefficient simplifies the problem while retaining its a solved status. Though not commonly employed for solving differential equations, this method creates a more favorable setting for tackling such problems. Apart from the fact that our equation (E) has the variable coefficient $p_0(x) \neq 0$, another advantage of this transformation lies in the ability to find solutions using eigenvalues, whose existence is closely related to the well-established theory known as *The Oscillation Theory* (Agarwal 2012, Koçi 2015). This theory originated in 1836 with the works of *Sturm-Liouville* (Byrne 2009, Hoxha 2023) and has since been extensively developed, with numerous books and studies contributing to its advancement. Its applications extend beyond the study of ordinary differential equations (Koçi 2009; Beqiri *et al.*,2013). Furthermore, recent research has applied the results of this theory to investigate oscillatory behavior in mathematical modeling. Drawing from the work of (Christodoulou *et al.*, 2016; Christodoulou *et al.*, 2021) and our previous research (Koçi 2022) on predicting the intervals of oscillations in the solutions of ordinary second-order linear homogeneous differential equations, this paper analyzes the interval of oscillation for the solutions of the *Gegenbauer differential equation* (Weisstein 2002; Al-Masaeed 2023).

Consider the following differential equation:

$$y'' + p(x)y = 0.$$

Definition 1 (Koçi 2015): A nontrivial solution $y(x)$ of the above equation is oscillatory if, for every number T, $y(x)$ has an infinite number of zeros over the open interval $(T, +\infty)$, such that for every $\beta > \alpha$, there exists a number $\xi > \beta$, such that $y(\xi) = 0$.

Definition 2 (Koçi 2015): A nontrivial solution $y(x)$ of the above equation is nonoscillatory if for any number T, $y(x)$ has a finite number of zeros over the open interval $(T, +\infty)$, such that there exists $\beta > \alpha$ such that $y(\zeta) \neq 0$, for every $\zeta > \beta$.

Theorem 1 (Ambrosetti 2015):
Let $y(x)$ and $z(x)$ be two nontrivial solutions of equations:

$$y'' + f(x)y = 0,$$
$$z'' + g(x)z = 0$$

for every x from an open interval (a, b), where $f(x) > g(x)$. Between two consecutive zeros x_1 and x_2 of $y(x)$, there exists at least one zero of $z(x)$ over (x_1, x_2).

The general form of second order linear homogeneous differential equation is:

$$y'' + b(x)y' + c(x)y = 0. \qquad (1)$$

One property of linear homogenous differential equation is that they remain the same even after substitution of unknown function y using the following relation:

$$y(x) = \alpha(x)z(x).$$

In the new equation we get that the coefficient near z' is:

$$2\alpha'(x) + b(x)\alpha(x).$$

We want this coefficient to be absent so it is enough to choose (x) such that

$$2\alpha'(x) + b(x)\alpha(x) = 0$$

$$|\alpha(x)| = exp\left[-\frac{1}{2}\int b(x)dx\right]$$

By using the following substitution:

$$y(x) = z(x)\,exp\left[-\frac{1}{2}\int b(x)dx\right]$$

the equation (1) is transformed into its canonical form:

$$z''(x) + q(x)z(x) = 0. \qquad (2)$$

In this case, the coefficient $q(x)$ will be:

$$q(x) = c(x) - \frac{b^2(x)}{4} - \frac{b'(x)}{2}.$$

Since every linear transformation of dependent and independent variable do not change the linearity of the equation, we use the following two methods to find oscillatory solution.

1. METHOD (Christodoulou *et al*., 2016; Koçi 2022; Hoxha 2023):

Transformation of dependent variable $z(x)$

The substitution of unknown function $z(x)$ by using the relation $z(x) = u(x)v(x)$ into equation (2) leads to

$$u''(x) + \frac{2v'(x)}{v(x)}u'(x) + \left[\frac{v''(x)}{v(x)} + q(x)\right]u(x) = 0. \qquad (3)$$

The coefficient before $u'(x)$ should be constant, let's consider to be c:

$$\frac{2v'(x)}{v(x)} = c$$

then $v(x) = e^{\frac{cx}{2}}$.

The equation (3) will have the following form:

$$u''(x) + cu'(x) + \left[\frac{c^2}{4} + q(x)\right]u(x) = 0$$
(4)

and its canonical form is

$$u''(x) + q_1(x)u(x) = 0.$$

The constant coefficient damping term $u'(x)$ will not appear in the coefficient $q_1(x)$ of canonical form of (4). For equation (4), its discriminants are calculated and:

$$d(x) = c^2 - 4\left[\frac{c^2}{4} + q(x)\right] = -4q(x),$$

where the constant c has been dropped out, and $q_1(x)$ is returned into $q(x)$, since

$$q_1(x) = -\frac{d(x)}{4} = q(x).$$

Therefore, this method does not work.

Transformation of independent variable x

The substitution of variable x by using the relation $x = \varphi(t)$ into equation (2), where $\varphi(t)$ is an arbitrary function, twice differentiable over an open interval $(\alpha, \beta) \subseteq (a, b)$, which corresponds to the change of x over the open interval (a, b) such that $\varphi'(t) \neq 0$, for every $t \in (\alpha, \beta)$. By simple calculations:

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{1}{\varphi'(t)},$$

$$\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2}\frac{1}{[\varphi'(t)]^2} - \frac{dy}{dt}\frac{\varphi''(t)}{[\varphi'(t)]^3}$$

By using this transformation, equation (2) has the following form

$$z'' - \frac{\varphi''}{\varphi'} z' + [\varphi'(t)]^2 q(\varphi(t)) z = 0, \qquad (5)$$

where the derivatives are related to the new variable t. In this equation, choose the function $\varphi(t)$ such that $\frac{\varphi''}{\varphi'} = k$, where k , is an arbitrary constant. The differential equation $\varphi'' - k\varphi' = 0$ is solved by reducing order, so its solution is:

$$x = \varphi(t) = c_1 + c_2 e^{kt}, \qquad (6)$$

where c_1 and c_2 are arbitrary constant. The equation (5) will have the following form:

$$z'' - kz' + k^2 (x - c_1)^2 q(x) z = 0. \qquad (7)$$

For equation (7), let's construct its canonical form by using the ensuing substitution:

$$z(t) = w(t)\, exp\left(\frac{1}{2} ct\right).$$

Its canonical form will be:

$$w'' + \hat{q}(x) w = 0 \qquad (8)$$

where:

$$\hat{q}(x) = k^2 \left[(x - c_1)^2 q(x) - \frac{1}{4}\right]. \qquad (9)$$

The transformation of the independent variable x allows for the introduction of constants k and c_1. The constant k^2 acts as a proportionality factor in the coefficient $q_1(x)$ of the canonical form. The constant c_1 represents a horizontal shift of independent variable x. Thus, by an appropriate choice of c_1, the shifted term $(x - c_1)^2$ in equation (9) can eliminate any regular singular point that the given term $q(x)$ may contain. In this case, the term $q(x)$ remains constant. The well-known comparison criterion for oscillatory solution is applied to equation (8), even though the coefficient before $w(t)$ depends on $x(t)$. The Sturm comparison theorem enables a comparison between equation (9) and a simple oscillatory harmonic equation:

$$y'' + \varepsilon^2 y = 0,$$

where the constant ε approaches to 0.

From Theorem 1, oscillatory solutions are obtained when:

$$(x - c_1)^2 q(x) - \frac{1}{4} > \varepsilon^2 \to 0,$$

so, within the open interval where x satisfies the following inequality:

$$q(x) > \frac{1}{4(x - c_1)^2}. \quad (10)$$

For many equations in applied mathematics, singularities in their coefficient occur when $x = 0$. In this case we obtain $c_1 = 0$, and the criterion takes the following form:

$$q(x) > \frac{1}{4x^2}. \quad (11)$$

2. RESULTS

The equation to be addressed has the following form:

$$(1 - x^2)y'' - (2\mu + 1)xy' + n(n + 2\mu)y = 0, \quad (12)$$

where, $|x| < 1$, μ is a random constant, and $n > 0$, and is called ultraspherical or *Gegenbauer differential equation.* (Weisstein 2002).
This equation is equivalent to:

$$y'' - \frac{(2\mu + 1)}{1 - x^2}xy' + \frac{n(n + 2\mu)}{1 - x^2}y = 0, \quad (13)$$

and is transformed into its canonical form (2), where the coefficient $q(x)$ is:

$$q(x) = c(x) - \frac{b^2(x)}{4} - \frac{b'(x)}{2}$$
$$= \frac{n(n+2\mu)}{1-x^2} - \frac{1}{4}\left[-\frac{(2\mu+1)x}{1-x^2}\right]^2 - \frac{1}{2}\left[-\frac{(2\mu+1)x}{1-x^2}\right]'$$
$$= \frac{n(n+2\mu)}{1-x^2} + \frac{2(2\mu+1) + x^2[2(2\mu+1) - (2\mu+1)^2]}{4(1-x^2)^2}.$$

$q(x)$ is an even function of x, so it is sufficient to search for oscillatory solutions only for $x \geq 0$, with generalization of this result for $x < 0$ due to symmetry of $q(x)$.

In the equation (6), we choose $c_1 = 1$ and $k = 1$, then this equation has the form:

$$x = 1 + c_2 e^t. \tag{14}$$

Then the criterion for oscillatory solutions becomes:

$$q(x) > \frac{1}{4(x-1)^2}.$$

By choosing $c_2 = \pm 1$ into equation (14), we search for oscillatory solutions on both sides of $x = 1$.

For $c_2 = -1$, then $1 - e^t \geq 0$, so $t \leq 0 \Rightarrow x \in [0,1)$, and for $c_2 = 1$, $1 + e^t \geq 0$, so for $\forall t \in R \Rightarrow x \in (1, +\infty)$.

The criterion is applicable in both cases. For $x = \pm 1$, it is not feasable to search for oscillatory solutions because if $x = \pm 1$, the equation is not a second-order differential equation, so x must be different from ± 1.

From criterion it is required that:

$$q(x) = \frac{4n(n+2\mu)(1-x^2) + (2\mu+1)(x^2 - 2\mu x^2 + 2)}{4(1-x^2)^2} > \frac{1}{4(x-1)^2}$$
$$\frac{4n(n+2\mu)(1-x^2) + (2\mu+1)(x^2 - 2\mu x^2 + 2)}{(1+x)^2} > 1.$$

Since $(1+x)^2 > 0$, then the following inequality holds:

$$4n(n+2\mu)(1-x^2)+(2\mu+1)(x^2-2\mu x^2+2)>(1+x)^2$$
$$4n^2+8n\mu-4n^2x^2-8n\mu x^2+2\mu x^2-4\mu^2x^2+4\mu-2\mu x^2+1-2x>0$$
$$-(2n+2\mu)^2x^2-2x+(2n+2\mu)^2-4\mu^2+4\mu+1>0.$$

Let's denote $N=(2n+2\mu)^2$ and $2M=-4\mu^2+4\mu+1$.

The above inequality can be rewritten as: $Nx^2+2x-(N+2M)<0$.

To solve this inequality, the discriminant should be positive to find solutions to previous inequality. Thus, we obtain:

$$_2x_1=\frac{-b\pm\sqrt{D}}{2a}=\frac{-1\pm\sqrt{1+N(N+2M)}}{N}.$$

The previous inequality $Nx^2+2x-(N+2M)<0$ has solution for $x\in(x_1,x_2)$, where $x_1<0$ and $x_2>0$ so, the oscillatory solutions are on the semi closed interval $[0,x_2)$.

Since $q(x)$ is an even function, *Gegenbauer differential equation* has oscillatory solutions on open interval $(-x_2,x_2)$.

For every value of μ, it is showed that $x_2\le 1$ and the oscillation interval of this equation becomes $(-1,1)$. For fixed values of μ, some well-known differential equation of mathematical physics such that *Chebyshev differential equation* of first type, *Legendre differential equation* etc. are obtained from the *Gegenbauer differential equation.*

By knowing that these equations are obtained as special cases of the *Gegenbauer differential equation*, it can be concluded also the solutions of these equations are oscillatory over the open interval $(-1,1)$.To visualize the oscillation intervals for the *Gegenbauer differential equation*, we used MatLab to solve this equation. The solutions of this differential equation are called *Gegenbauer polynomials*, and below, we treat as an example the case when $n=4$ and $\mu\in\{1,2,3,4,5\}$.

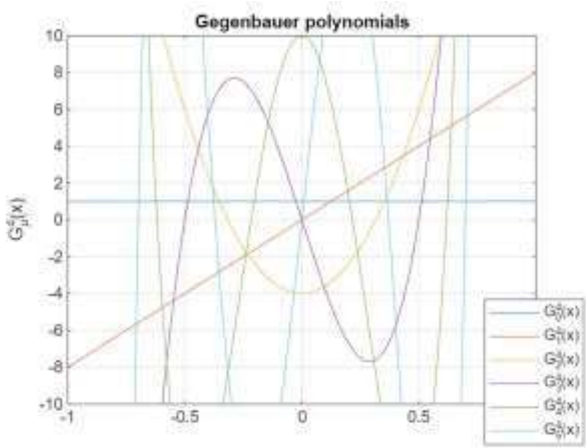

Fig.1: Gegenbauer polynomials for $n = 4$ and $\mu \in \{1,2,3,4,5\}$.

4. CONCLUSIONS

In this paper, we have presented a methodology for predicting the intervals of oscillation of solutions of second-order linear homogenous differential equations, based on the behavior of their variable coefficients. However, within these intervals, the oscillatory behavior of solutions does not conform to the classical behavior of oscillation theory, which typically entails the existence of solutions with an infinite number of zeros. Instead, the uniqueness of oscillation in this context encompasses a broader spectrum of situations, including the situation associated with the appearance of consecutive critical points of the same type (maximum, minimum, or inflection points) in the graph of a solution, which we now consider as oscillatory solutions. Equations or special functions arising from mathematical physics and beyond can be studied with this method in the intervals where they exhibit oscillatory properties of their solutions. To validate the applicability of this criterion for oscillatory solutions, we have detailed how to find the oscillation interval for the *Gegenbauer differential equation.*

5. RECOMMENDATIONS

The authors suggest further study on the application of this method to generalized equations derived from physics problems. Additionally, they propose investigating whether a generalized method exists, where the treatment presented in this paper can be viewed as a special case of this generalization.

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