HARMONIZING DOMINANCE: EXPLORING THE ORDER MCSHANE AND THE ORDER WEAK-MCSHANE INTEGRALS IN BANACH LATTICES

Mimoza Shkëmbi
Department of Mathematics, University of Elbasan, Albania
John Shkëmbi
Department of Electrical Engineering and Computer Science, USMA, West Point, USA

ABSTRACT

It is widely recognized that the McShane integral of real-valued functions, a Riemann-type integral, is equivalent to the Lebesgue integral. Building on this foundational equivalence, McShane's unique approach to Lebesgue's integration theory has since been extended to the domain of vector-valued functions. For functions taking values in a Banach space, the integral of choice is the Bochner integral. This integral is known to be stronger than the Birkhoff integral, and the Birkhoff integral is stronger than the McShane, Pettis, and Dunford integrals. Following this, the development of order-type integrals has emerged, especially for functions valued in ordered vector spaces, and more specifically, in Banach lattices, which accommodate a structured approach to integration based on the order of elements. In this paper, we introduce an alternative definition of the weak McShane integral and explore McShane-type integrals within the realm of functions defined on a compact metric Hausdorff topological space (S, d), where these functions take values in a Banach lattice with an order-continuous norm, denoted as X. Our investigation encompasses both the norm and the order structure of the space X. Throughout this study, we derive several noteworthy outcomes that illuminate the relationships between these integrals. This is particularly evident in scenarios where the Banach space X does not contain any copy of c_0, while proving the equivalence of the weak order-type McShane and order Pettis' integrals.

Keyword(s): (oP)-integral, (oM)-integral, (WoM)-integral
***Mathematics Subject Classification**:* 28B05, 28B20, 46B42.

1. INTRODUCTION

Within the domain of functions valued in Banach spaces, the hierarchy of integrals places the Bochner integral at a more robust position compared to the McShane integral, which itself surpasses the Henstock and Pettis integrals in strength. This hierarchical structuring of integrals has been rigorously delineated in the scholarly corpus, with seminal contributions being acknowledged from

Bukhvalov et al. (1979), Meyer-Nieberg (1991), Cao (1992), Fremlin (1995, 2002), Di Piazza and Marraffa (2002), Kurtz and Swartz (2004), Schwabik and Guoju (2005), Candeloro and Sambucini (2014), Di P iazza and Musiał (2014).

Furthermore, the exploration of order-type integrals within ordered vector spaces, especially Banach lattices, has led to notable advancements. Significant contributions to this area include the works of Luxemburg and Zaanen (1971), which laid the groundwork, followed by the studies of Candeloro (1996), Candeloro and Sambucini (2014), and the innovative approaches by Boccuto, Minotti, and Sambucini (2013) as well as Boccuto, Candeloro, and Sambucini (2014). Recent explorations into the order-type integrals by Candeloro and Sambucini (2015), and the studies conducted by Candeloro et al. (2018), continue to enrich the discourse, providing a comprehensive understanding of the distinctions between various integral types, particularly in the context of L-space-valued functions.

In this paper, we draw inspiration from both foundational and contemporary examinations to compare the order-McShane integral with other order-type integrals in these spaces, thereby advancing into new and independent directions of research. We observe that in the case of L-spaces,

the order-Pettis integral exhibits greater strength than the Bochner integral. Furthermore, we demonstrate the equivalence of the weak order-McShane and order-Pettis integrals within Banach lattices Y, specifically those that

contain no copy of c_0.

In the subsequent sections, we adopt the notation S to denote a compact

metric space, and μ to denote a regular, nonatomic, and σ- additive measure mapping from the σ-algebra $\mathfrak{B}$ of Borel sets of S to $\mathbb{R}_0^+$, such that $\mu : \mathfrak{B} \rightarrow \mathbb{R}_0^+$. The sequence $(p_n)_n$ is considered order-convergent ((*o*)-convergent) to p if a sequence $(v_n)_n \in \mathbb{R}$ exists such that $v_n \downarrow 0$ and $|p_n - p| \leq v_n$, $\forall\, n \in \mathbb{N}$ (Luxemburg et al., 1979; Candeloro and Sambucini, 2013). This is denoted as (o)-lim $_n p_n = p$. A gauge is any map $\delta: S \rightarrow \mathbb{R}^+$. On the other hand, a partition of S is a finite collection of pairs $\Pi = \{(A_i, \xi_i), \ldots., (A_i, \xi_r)\}$ where the sets A_i are non-overlapping and $\bigcup_{i=1}^{r} A_i = S$, and the points ξ_i are called tags. A partition is said to be of Henstock type (H-partition) if each tag ξ_i belongs to its corresponing set A_i; otherwise, it is either a free or McShane partition (M-partition). A gauge is considered δ-*fine* if the distance between any point τ $\in A_i$ and its corresponding tag ξ_i is less than $\delta(\xi_i)$ for all *i* = 1, . . . , *r*. An alternative definition of a gauge is a function that associates each point $\xi_i \in S$ covering the set A_i and with an open ball centered at ξ_i.

DEFINITION 1.1 (Candeloro and Sambucini, 2015)

A function $g : S \rightarrow Y$ is said to be order McShane integrable (order Henstock integrable) on S if there exist an $L \in Y$ such that for every (o)-sequence $(p_n)_n$ in Y, there is a corresponding sequence $(\delta_n)_n$ of gauges $(\delta_n(t) : S \rightarrow]0, +\infty[\,)$ such that for every n and (δ_n)-fine M-partition (H-partition) $\{(A_i, \xi_i), \ldots., (A_i, \xi_r)\}$ of S, the following inequality holds:

$$|\textstyle\sum_{\Pi} g - L| \leq p_n,$$

where $\sum_{\Pi} g = \sum_{i=1}^{r} g(\xi_i)\mu(A_i)$. We write $L = (oM)\int_S \, g \, d\mu$ (respectively, $L = (oH)\int_S \, g \, d\mu$), and L is (o)-McShane integral ((o)- Henstock integral) of g on S. The function g is (oM)-integral ((oH)-integral) on a subset $I \subset S$ if the function $g \, . \, \chi_I$ is (oM)-integrable ((oH)-integral) on S, where χ_I is the characteristic function of the subset I. The (oM)-integral ((oH)-integral) of g on subset I is denoted as

$$(oM)\int_I \, g \, d\mu \, , ((oH)\int_I \, g \, d\mu).$$

Due to the (o)-continuity of the norm in Y, it follows that any map g, which is integrable in the sense of McShane (Henstock), also possesses McShane (Henstock) integrability.

DEFINITION 1.2.

The function $g : S \to Y$ is called order measurable (o-measurable) if, for any sequence $(p_n)_n$ with $p_n \downarrow 0$, there exists a sequence of simple functions $\{g_n\}_n$ such that
$|g_n(x) - g(x)| < p_n$ for almost all $x \in S$ and for all $n \in N$.
Let W denote the set of all (o)-McShane integrable functions g: $S \to Y$.

DEFINITION 1.3. (Shkëmbi *et al*., 2023)

A subset $G \subset W$ is said to be (oM)-equiintegrable ((oH)-equiintegrable) if, for any (o)- sequence $(p_n)_n$, there exists a corresponding sequence $(\delta_n)_n$ of gauges such that, for any $g \in G$, the following inequalities hold provided $\{(A_i, \xi_i), \ldots, (A_i, \xi_r)\}$ is a (δ_n)-fine M-partition (H-partition) of S:

$$\left|\sum_{i=1}^{r} g(\xi_i)\,\mu(A_i) - (oM)\int_S \, g\right| \leq p_n,$$
$$\left|\sum_{i=1}^{r} g(\xi_i)\,\mu(A_i) - (oH)\int_S \, g\right| \leq p_n.$$

PROPOSITION 1.4.
A function $g: S \to Y$ is (o)-McShane integrable if and only if the set $\{y^*(g); y^* \in B(Y^*)\}$ is (oM)-equiintegrable.

Proof. Suppose g is (oM)-integrable. Then, for any (o)-sequence $(p_n)_n$, there exists a corresponding sequence $(\delta_n)_n$ of gauges $(\delta_n(\xi): S \to]0, +\infty[\,)$ for every n, such that

$$\left|\sum_{i=1}^{r} g(\xi_i)\mu(A_i) - (oM)\int_S g\right| \leq p_n$$

F
or every δ_n-fine M-partition $\{(A_i,\xi_i),\ldots,(A_i,\xi_r)\}$ on S. For an arbitrary $y^* \in Y^*$, we have

$$\left|\sum_i y^*\big(g(\xi_i)\big)\mu(A_i) - y^*((oM)\int_S g)\right|$$
$$\leq |y^*| \cdot \left|\sum_i g(\xi_i)\mu(A_i) - (oM)\int_S G\right| \leq p_n \cdot sup|y^*|.$$

This shows that $\{y^*(g); g^* \in B(Y^*)\}$ is (oM)-equiintegrable.

Conversely, assume that $\{y^*(g); g^* \in B(Y^*)\}$ is (oM)-equiintegrable. Then, for every (o)- sequence $(p_n)_n$, there exists a corresponding sequence $(\delta_n)_n$ of gauges $(\delta_n(\xi): S \to]0, +\infty[\,)$ for every n on S, such that

$$\left|\sum_i y^*\big(g(\xi_i)\big)\mu(A_i) - (oM)\int_S y^*(g)\right| \leq p_n$$

for every δ_n-fine M-partition $\{(A_i,\xi_i),\ldots,(A_i,\xi_r)\}$ and $\{(I_i,t_i), i = 1,\ldots,p\}$ on S and $y^* \in B(Y^*)$. Hence, if $\{(A_i,\xi_i)\}, \{(E_j,\eta_j)\}$ are δ_n-fine M-partition of S, we get:

$$|y^*(\textstyle\sum_i g(\xi_i)\mu(A_i) - \sum_j g(\eta_j)\mu(E_j))|$$
$$= \left|\sum_i y^*\big(g(\xi_i)\big)\mu(A_i) - \sum_j y^*\big(g(\eta_j)\big)\mu(E_j)\right| \leq 2p_n$$

for every $y^* \in B(Y^*)$. Hence,

$$|\Sigma_i g(\xi_i)\mu(A_i) - \Sigma_j g(\eta_j)\mu(E_j)| \leq 2p_n$$

and the function g is (oM)-integrable.

2. MATERIALS AND METHODS

Our study commences by exploring the weak order McShane integral and the order Pettis integral within the context of Banach lattices. We aim to provide insightful comparison results in this specific setting.

DEFINITION 2.1.

A function $g : S \to Y$ is termed order Bochner integrable if there exists an (o)-Cauchy sequence $\{g_n\}_n$ of simple functions that converges in measure to g almost everywhere in S. In other words,

$$(o)\text{-lim} \int_S |g_n(\xi) - g_m(\xi)| = 0 \text{ for almost all } \xi \in S.$$

The limit

$$(o)\text{-}\lim_{n\to\infty} \int_S g_n(\xi)$$

is referred to as the order Bochner integral of the function g, denoted as $(oB) \int_S g(\xi)$. Here, g_n is an arbitrary sequence of simple functions that determines g.

DEFINITION 2.2.

If $g : S \to Y$ is weakly (o)-measurable, and for each $y^* \in Y^*$, the function $y^*(g) : S \to \mathbb{R}$ is (oM)-integrable, then g is called order Dunford integrable. The order Dunford integral $(oD) \int_I g$ of g over a measurable set $A \subset S$ is defined by the element $y_A^{**} \in Y^{**}$, denoted as

$$(oD) \int_A g = y_I^{**} \in Y^{**}.$$

For all $y^* \in Y^*, y_A^{**}(y^*) = \int_A y^*(g)$.

The Dunford integral $(oD)\int_A g$ is an element of the second dual Y^{**} of the Banach lattice Y. However, it is worth noting that this situation might be less preferable, as one would expect that the values of an integral of an Y-valued function to belong to the same space Y. The space Y itself is naturally embedded into Y^{**}. If $(oD)\int_I g \in Y \subset Y^{**}$, then the following alternative definition can be presented.

DEFINITION 2.3.

If $g: S \to Y$ is (oD)-integrable, where $(oD)\int_I g \in e(Y) \subset Y^{**}$, (with e being the canonical embedding of Y into Y^{**}) for every measurable $I \subset S$, then g is termed order Pettis integrable. The expression

$$(oP)\int_e g = (oD)\int_I g$$

is referred to as the (o)-Pettis integral of $g: S \to Y$ over the set I. The concept of (oP)-integrability can also be expressed equivalently as outlined in Definition 2.4.

DEFINITON 2.4.

A weakly measurable function $g: S \to Y$, where $y^*(g)$ is (o)-Bochner integrable for every $y^* \in Y^*$, is considered (o)-Pettis integrable if, for every measurable $I \subset S$, there exists an element $y_I \in Y$ such that,

$$y^*(y_I) = \int_I y^*(g)$$

holds for every $y^* \in Y^*$. In the case where Y is a reflexive space $(Y^{**} = Y)$, the (oD) and (oP) integrals coincide.

THEOREM 2.5. (Candeloro and Sambucini 2015)

Let $g: S \to Y$ be (oM)-integrable and assume that Y is an L-space. Then, g is Bochner integrable.

PROPOSITION 2.6.

If $g : S \to Y$ is (oB)-integrable, then g is (oP)-integrable:

$$(oP) \int_E g = (oB) \int_E g$$

for every measurable $A \subset S$.

Proof. Since $G \in (oB)$, let (G_q) be an (o)-Cauchy sequence of simple functions determining g. Then,

$$(oB) \int_A g = {}^{o-}\lim_{q\to\infty}(oB) \int_A g_q$$

For $y^* \in Y^*$, we have:

$$y^*\left((oB) \int_A g\right) = Y^*\left({}^{o-}\lim_{q\to\infty}(oB) \int_A g_q\right) = {}^{o-}\lim_{q\to\infty} y^*\left((oB) \int_A g_q\right)$$
$$= {}^{o-}\lim_{q\to\infty}(oB) \int_A y^*(g_q) = \int_A y^* g,$$

because

$$\left|(oB) \int_A y^*(g_q - g)\right| \le (oB) \int_A |y^*(g_q - g)|$$

$$\le (oB) \int_A sup|y^*(g_q - g)| \le \|y^*\|(oB) \int_A |g_q - g|,$$

and

$$= {}^{o-}\lim_{q\to\infty}(oB) \int_A |g_q - g| = 0.$$

Hence, $g \in (oP)$.

PROPOSITION 2.7.

If $g : S \to Y$ is (oM) integrable with $(oM) \int_S g \in Y$, then for every $y^* \in Y^*$, the real function $y^*(g) : S \to \mathbb{R}$ is McShane integrable, and

$$(oM)\int_S y^*(g) = y^*\left((oM)\int_S g\right).$$

Proof. By Definition 1.1, for every $(\boldsymbol{o})$-sequence $(\boldsymbol{p_n})_n$ in Y, there is a corresponding sequence $(\boldsymbol{\delta_n})_n$ of gauges $(\boldsymbol{\delta_n(t)} : \boldsymbol{S} \to]\mathbf{0}, +\infty[\,)$ such that for every $\boldsymbol{n}$ and $(\boldsymbol{\delta_n})$-fine M-partition $\{(\boldsymbol{A_i}, \boldsymbol{\xi_i}), \boldsymbol{i} = \mathbf{1}, \dots, \boldsymbol{r}\}$ of $\boldsymbol{S}$, the inequality

$$\left|\sum_{i=1}^{r} g(\xi_i)\,\mu(A_i) - (oM)\int_S G\right| \le P_n$$

holds. If $y^* \in Y^*$, then by the previous inequality, we have

$$\left|\sum_{i=1}^{r} y^*\left(g(\xi_i)\mu(A_i) - y^*(oM)\int_S g\right)\right|$$
$$=\left| y^* \sum_{i=1}^{r}\left(g(\xi_i)\mu(A_i) - (oM)\int_S g\right)\right|$$
$$\le \|y^*\| \left|\sum_{i=1}^{r} f(\xi_i)\,\mu(A_i) - (oM)\int_S g\right| \le \|y^*\|.p_n \ ,$$

for every n and (δ_n)-fine M-partition $\{(A_i, \xi_i), i = 1, \dots, r\}$ of S. The same holds if (oM) is replaced by (oH), and H-partitions are used instead of M-partitions.

REMARK. The (oM), (o)-Bochner (Lebesgue) integrals of $y^*(g) : S \to \mathbb{R}$ coincide $(Y = \mathbb{R})$, and therefore, in Proposition 2.7, we can replace the (oM)-integrability of $y^*(g) : S \to \mathbb{R}$ with its (o)-Bochner (Lebesgue) integrability. Consequently, we also have that the function $g : S \to Y$ is weakly measurable.

THEOREM 2.8.

If $g : S \to Y$ is (oM)-integrable with $(oM)\int_S g \in Y$, then g is also (oP)-integrable, and

$$(oP)\int_E g = (oM)\int_S g\,\chi_E = (oM)\int_E g$$

For every measurable $E \subset S$. Hence, we have $(oM) \subset (oP)$.

Proof. According to the previous remark, the function $g: S \to Y$ is weakly measurable. For every measurable set $E \subset S$, the function $g.\chi_E$ is McShane integrable, and

$$(oM)\int_S f.\chi_E = (oM)\int_E g \in Y.$$

Hence, by Proposition 2.7, for every $y^* \in Y^*$, the real function $y^*(g.\chi_E)$ is (oM)-integrable, and

$$(oM)\int_S y^*(g.\chi_E) = (oM)\int_E y^* g = y^*\left(oM\int_E g\right).$$

By Definition 2.4, this implies that g is (oP)-integrable.

3. RESULTS

THEOREM 3.1.

If $g_u: S \to Y,\ u \in N$, are (oM)-integrable functions such that:

1. $g_u(\xi) \to g(\xi)$ for $\xi \in S$,
2. the set $\{g_u; u \in N\}$ forms an (oM)-equiintegrable sequence.

Then $g_u \cdot \chi_E,\ u \in N$, is an (oM)-equiintegrable sequence for every measurable set $E \subset S$.

Proof. For every (o)-sequence $(p_n)_n$, there exists $\eta > 0$ such that, assuming $E \subset S$ is measurable, there exist $F \subset S$ closed and $G \subset S$ open, such that, $F \subset E \subset G$ and $\mu(G \setminus F) < \eta$. Assume the sequence $(\delta_n)_n$ corresponds to the gauge: $\delta_n : T \to]0, \infty[$ such that,

$$B(\xi, (\delta_n(\xi) \subset G \text{ for } \xi \in G,$$
$$B(\xi, (\delta_n(\xi) \cap S \subset S \setminus F \text{ for } \xi \in S \setminus F,$$

and that $\{(A_i,\xi_i)\}$ and $\{(L_j,\zeta_j)\}$ are δ_n-fine M-partitions of S.

1. If $\xi_i \in E$, then $A_i \subset G$, and $F \subset \mathrm{int}\left(\bigcup_{\xi_i\in F} A_i\right)$.
2. If $\zeta_j \in E$, then $L_j \subset G$, and $F \subset \mathrm{int}\left(\bigcup_{\zeta_j\in F} L_j\right)$.
3. $\left|\sum_{i,\xi_i\in A} g_u(\xi_i)\mu(A_i) - \sum_{j,\zeta_j\in A} g_u(\zeta_j)\mu(L_j)\right| \leq p_n$.

Therefore, also

4. $\left|\sum_i g_u(\xi_i)\chi_E(\xi_i)\mu(A_i) - \sum_j g_u(\zeta_j)\chi_E(\zeta_j)\mu(L_j)\right| \leq p_n$

This is the Bolzano-Cauchy condition for (oM)-equiintegrability of the sequence $g_u \,.\, \chi_E$, $u \in N$.

THEOREM 3.2.

Let $g : S \to Y$ be a measurable function. If g is (oP)-integrable on S, then g is also (oM)-integrable on S.

Proof. The measurability of g implies that there is a bounded measurable function $q : S \to Y$ and a measurable function $h : S \to Y$ defined as

$$\mathrm{h}(\xi) = \sum_{n=1}^{\infty} x_n \chi_{E_n}(\xi),\ \xi \in S,$$

where $n \in \mathbb{N}$, $x_n \in Y$ and $E_n \subset S$ being pairwise disjoint measurable sets, we can express

$$g(\xi) = q(\xi) + h(\xi),\ \xi \in S.$$

The function q is bounded and measurable, making it (oB)-integrable since S is assumed to be a compact interval. Therefore, q is (oM)-integrable by Theorem 3.1, and by Theorem 2.8, it is also (oP)-integrable. Since f is assumed to be (oP)-integrable, the function $h = g - q$ must be (oP)-integrable. Consequently, the series $\sum_{n=1}^{\infty} x_n \mu(E_n)$ converges unconditionally in Y. The sequence $h_n(\xi) = \sum_{j=1}^{n} x_j \chi_{E_j}(\xi)$, $\xi \in S$, $n \in \mathbb{N}$, is (o)-McShane equiintegrable, and as such, it is certain to see that

$$\lim_{n\to\infty} h_n(\xi) = h(\xi) \text{ for } \xi \in S.$$

Then, the function h is (oM)-integrable:

$$(oM)\int_S h = \lim_{n\to\infty}(oM)\int_S h_n = \lim_{n\to\infty} \sum_{j=1}^{n} x_j\mu(E_j) = \sum_{j=1}^{\infty} x_j\mu(E_j)$$

Therefore, g is also (oM)-integrable.

THEOREM 3.3.

Assume that the Banach lattice Y is separable, and let $g : S \to Y$ be (oP)-integrable. Then, g is also (oM)-integrable.

Proof. The (oP)-integrability of g assumes that g is weakly measurable, and the function g is measurable. Theorem 3.2 establishes the (oM)-integrability of g.

Using Theorem 3.2 and Theorem 3.3, we immediately obtain the following result.

Corollary 3.4. Assume that the Banach lattice Y is separable. Then, $g : S \to Y$ is (oP)-integrable if and only if it is (oM)-integrable; formally, (oM) =(oP) holds in this case.

DEFINITION 3.5

An Y-valued function g is said to be weakly (oM)- integrable on S if, for every $y^* \in Y^*$, the real function $y^*(g)$ is (oM)-integrable on S. Additionally, for every interval $A \subset S$, there exists $y_A \in Y$ such that:

$$(oM)\int_A y^*(g) = y^*(y_A).$$

We denote $Y_A = (WoM)\int_A g$. A function g is weakly (oM)-integrable on a set $E \subseteq S$ if the function $g\chi_E$ is weakly (oM)-integrable on S, where χ_E denotes the characteristic function of E.

We write

$$(WoM)\int_S f.\chi_E = (WoM)\int_E g \in Y.$$

for the weak (oM)-integral of g on E.

THEOREM 3.6.

Suppose that Y is a Banach lattice, and let $g : S \to Y$ be a given function. Then, the weak (oM)-integral and the (oP)-integral of the function g are equivalent if and only if Y contains no copy of c_0.

Proof. If the weak (oM)-integral and the (oP)-integral are equivalent, according to Example 25 (Schwabik and Guoju, 2014), it is evident that Y cannot contain a copy of c_0. Conversely, assume that g is weakly (oM)-integrable on S, and Y contains no copy of c_0.

Suppose E is a measurable subset of S. Given γ such that $0 < \gamma < 1$, an interval A is called γ-regular if,

$$\tau(A) = \frac{\mu(A)}{d(A)} > \gamma,$$

where $\tau(A)$ is the regularity of the interval A, and $d(A) = \sup\{|x - y|;\ x, y \in A\}$. For every (o)-sequence $(p_n)_n$ in Y, there is a corresponding sequence $(\delta_n)_n$ of gauges $(\delta_n(t) : S \to]0, +\infty[)$.

Let $0 < \gamma < 1$ be fixed. Set $\Omega_n = \{A \subset S, A$ is an interval; $\xi \in A \subset B(\xi, (\delta_n(\xi)), \tau(A) > \gamma, \xi \in E\}$, where $\Omega = \{\Omega_n;\ n \in N\}$ is a Vitali cover of E.

By the Vitali covering theorem, there is a sequence E_n, where E_n is the finite union of non-overlapping intervals belonging to Ω such that $\mu(E \setminus \bigcup_n A_n) < p_n$. Hence, there exists a sequence A_n of non-overlapping intervals A_n such that $\mu(E \setminus \bigcup_n A_n) = 0$. Because g is weakly (oM)-integrable, the real function $y^*(g)$ is (oM)-integrable, and $(WoM)\int_{A_n} f \in Y$ for all $n \in N$.

Therefore, for each $y^* \in Y^*$, $(oM)\int_{E\setminus \bigcup_n A_n} y^*(g) = 0$:

$$(oM)\int_E \ y^*(g) = (oM)\int_{E\setminus \bigcup_n A_n} y^*(g) + (oM)\int_{\bigcup_n A_n} y^*(g) =$$
$$(oM)\int_{\bigcup_n A_n} y^*(g) =$$
$$\sum_n (oM)\int_{A_n} y^*(g) = \sum_n y^*((WoM)\int_{A_n} g) =$$
$$(o) - \lim_{r\to\infty} \sum_{n=1}^{r} y^*((WoM)\int_{A_n} g).$$

Since Y contains no copy of c_o, by the Bessaga-Pelczynski Theorem (Bukhvalov, *et al.*, , 1979, p. 22), the series $\sum_n (WoM)\int_{A_n} g$ is unconditionally convergent in norm to an element $y_E \in Y$,

$$(\text{o}) - \lim_{r\to\infty} \left| \sum_{n=1}^{r} (WoM\int_{A_n} g - y_E \right| = 0.$$

Since

$$y^*(\sum_{n=1}^{r}(WoM\int_{A_n} g - y_E) = \sum_{n=1}^{r} y^*((WoM)\int_{A_n} g) - y^*(y_E) \to 0$$

for $r \to \infty$, we obtain

$$(oM)\int_E \ y^*(g) = (\text{o}) - \lim_{r\to\infty} y^*(\sum_{n=1}^{r}(WoM\int_{A_n} g = y^*(y_E).$$

By definition, g is order Pettis integrable on S and $(WoM \int_{A_n} g = (oP) \int_{A_n} g$.

4. DISCUSSION

The established equivalence between weak order McShane integration and the order Pettis integral, particularly in the absence of a copy of c_0 emphasizes the distinction between different notions of integrability. This result not only deepens our theoretical understanding of these integrals in the but also raises questions about the underlying structures that influence their relationships. It prompts further exploration into the significance of Banach space properties, particularly the absence of c_0, in shaping the convergence behaviors of these integrals.

Moreover, the implications of these findings extend to broader applications in functional analysis, suggesting avenues for the development of integration theories and their potential impact on mathematical analysis more broadly. As the discourse on integrability progresses, these insights offer a substantive contribution to the scholarly dialogue regarding the interrelations among distinct integral frameworks, thereby facilitating the foundation for future scholarly inquiries into the terrain of mathematical integration.

5. CONCLUSION

Our investigation into McShane (weak McShane) integrability within the context of a metric compact regular space and a Banach lattice with an order-continuous norm has illuminated compelling connections with the order Pettis integral. Notably, we have uncovered a significant result: when the Banach space lacks a copy of c_0, the weak order McShane integral and the order Pettis integral become equivalent. This equivalence, contingent upon the absence of c_0, clarifies a distinctive condition under which these integrals align harmoniously. These discoveries not only augment the theoretical framework of integrability but also herald new avenues for in-depth investigation and application across the broader spectrum of functional analysis and the theoretical edifice of integration theory.

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