APPLICATION OF INCOMPLETE WEBER INTEGRALS IN DIFFRACTION PROBLEMS

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ABSTRACT

The present paper reports about the application of a class of incomplete cylindrical functions including incomplete functions of Bessel, Struve, and in particular the incomplete Weber exponential integrals in diffraction problems. It is shown that application of incomplete cylindrical functions in the diffraction of Gaussian beam by a system of equiareal circular sectors leads to a correct solution for the complex amplitude and the irradiance of the diffraction pattern. The diffraction pattern is observed on a parallel plane screen separated by a distance z from the plane of the obstacle. The problem is treated in polar coordinates r, θ on the plane of the diffraction obstacle, and ρ, φ on the plane where the diffraction pattern is observed. Mathematical analysis shows fact that the diffractional intensity distribution represents a system of ellipses with the principal semi-axis parallel to the edge of the obstacle.

***Keywords*:** incomplete cylindrical functions, diffraction, Gaussian beam

1. INTRODUCTION

The Fresnel-Kirchhoff scalar diffraction theory is applied with appropriate approximations and modifications to solve the problem of diffraction by regular-shape apertures such as single-slit or circular aperture and diffraction on a half plane. This problem addresses the diffraction of a plane wave by such apertures, i.e., mathematically concerning the study of solutions of the two-dimensional wave equation which obey to the Dirichlet boundary conditions or Neumann boundary conditions on the edges of apertures.

Difficulties appear when dealing with diffraction on apertures of more complicated shapes where, in particular, cylindrical waves are involved. Here, many optical sources do not produce beams that can be approximated by a uniform plane wave. Therefore, it is of interest to consider modifications to the plane-wave diffraction theory appropriate to non-uniform beams, seeking solutions using incomplete cylindrical functions. In addition, the unique phase and spatial-amplitude variation which is characteristic of lasers makes them ideal sources for diffraction experiments dealing with beam-size effects.

2. MATHEMATICAL CONSIDERATIONS

Let us first consider some properties of general incomplete cylindrical functions expressed in Poisson form. In any of the integral representations for cylindrical functions, it is sufficient to perform the integration over only a portion of the contour. Any function of the form:

$$f_\nu(z) = z^\nu \int e^{izt}\left(t^2-1\right)^{\nu-\frac{1}{2}} dt$$

in which the integration is performed along an arbitrary contour represents, up to a constant factor, incomplete cylindrical functions (Watson 1944). We need to construct these functions in such a manner that, for limiting cases, they reduce to cylindrical functions.

Let us consider a function of the form:

$$\mathcal{E}_\nu^{(1)}(c,z) = \frac{\Gamma\left(\frac{1}{2}-\nu\right)}{i\pi\Gamma\left(\frac{1}{2}\right)}\left(\frac{z}{2}\right)^\nu \int_c^{(1+)} e^{izt}\left(t^2-1\right)^{\nu-\frac{1}{2}} dt \tag{1}$$

where ν , c, and z are arbitrary complex numbers. We can select the contour of integration as shown in Figure 1 by the solid line. It begins at the point C in the complex t-plane, moves around the branch point $t=1$ and returns to the original point C. For definiteness we will assume that, as the contour is traversed, the argument of $\left(t^2-1\right)$ is zero at the point A and is equal to -2π at the point B. The resulting function $\mathcal{E}_\nu^{(1)}(c,z)$ is analytic in all the variables (Agrest and Maksimov 1972).

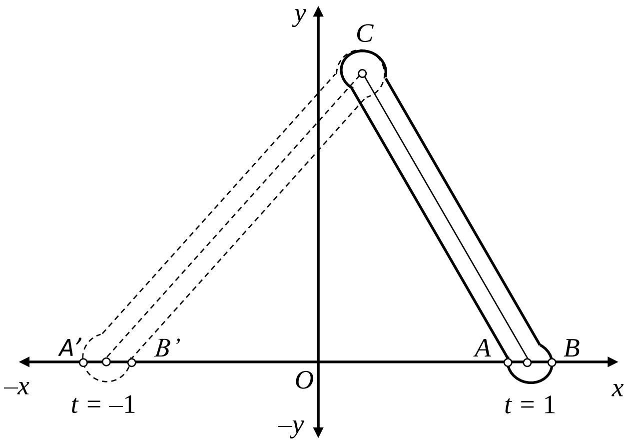

Fig. 1: Integration contours for the incomplete cylindrical function of Poisson form.

In analogy, we can introduce another function $\mathcal{E}_{\nu}^{(2)}(c,z)$, in which the contour is shown by the dotted line (Figure 1).

It can be verified that the sum of these functions is the Bessel function $J_{\nu}(z)$. Thus, it is sufficient to consider only the function $\mathcal{E}_{\nu}^{(1)}(c,z)$. Rearranging the terms, we obtain in the *w* plane a function of the following form:

$$E_{\nu}^{\pm}(w,z)=\frac{2z^{\nu}}{A_{\nu}}\int_{0}^{w} e^{\pm iz\cos u}\sin^{2\nu} u du \qquad (2)$$

For $w=\pi$, the right hand side is none other than the Bessel function in Poisson form. For $w\neq\pi$, on the other hand, the analytic function $E_{\nu}^{\pm}(w,z)$ is incomplete cylindrical function of the Poisson form.

It is also convenient to represent (2) in the following form:

$$E_{\nu}^{\pm}(w,z)=J_{\nu}(w,z)\pm iH_{\nu}(w,z) \qquad (3)$$

where for $w=\pi/2$, the right hand side represents the familiar expressions for Bessel and Struve functions:

$$J_\nu(z) = \frac{2z^\nu}{A_\nu} \int_0^{\pi/2} \cos(z\cos\theta)\sin^{2\nu}\theta d\theta \tag{4}$$

$$H_\nu(z) = \frac{2z^\nu}{A_\nu} \int_0^{\pi/2} \sin(z\cos\theta)\sin^{2\nu}\theta d\theta \tag{5}$$

Therefore, $J_\nu(w,z)$ and $H_\nu(w,z)$ are the incomplete Bessel function and the incomplete Struve function, respectively.

Now, let us define the Weber exponential integrals. These integrals are improper integrals of incomplete cylindrical functions with weighting function $x^\lambda e^{-p^2x^2}$:

$$\int_0^\infty e^{-p^2x^2} E_\nu^+(\alpha, bx) x^\lambda dx \tag{6}$$

Integrals of the form (6) may also be reduced to known functions. If $\lambda + \nu = 2k$ is an integer, then this integral can be evaluated by means of the incomplete modified confluent hypergeometric function:

$$F_\nu^-(w,z) = \frac{1}{2} e^{-i\nu\frac{\pi}{2}} E_\nu^+(w, iz) = \frac{z^\nu}{A_\nu} \int_0^w e^{-z\cos\theta} \sin^{2\nu}\theta d\theta \tag{7}$$

yielding:

$$\int_0^\infty e^{-p^2x^2} J_0(\alpha, bx) dx = \frac{\sqrt{\pi}}{2p} e^{-\frac{b^2}{8p^2}} F_0^-\left(2\alpha, \frac{b^2}{8p^2}\right) \tag{8}$$

Setting $\alpha = \pi/2$ and recalling that (Agrest and Maximov 1972):

$$J_0\left(\frac{\pi}{2}, bx\right) \equiv J_0(bx) \quad \text{and} \quad F_0^-(\pi, z) = I_0(z)$$

we obtain the Weber integral for Bessel functions [Watson 1944, p.432]:

$$\int_0^\infty e^{-p^2x^2} J_0(bx)\,dx = \frac{\sqrt{\pi}}{2p}\exp\left[-\frac{b^2}{8p^2}\right] F_0^-\left(\frac{b^2}{8p^2}\right) \tag{9}$$

which we employ in the problem of diffracted light by a set of circular sectors.

3. COMPLEX AMPLITUDE OF DIFFRACTED LIGHT BY A SET OF CIRCULAR SECTORS

We investigate the behavior of a Gaussian beam produced by a laser source providing a Gaussian profile, passing through a grid containing N equal transparent circular sectors. This problem is solved with incomplete cylindrical functions followed by the solution of incomplete Weber exponential integrals.

Suppose that a monochromatic beam of light is incident on an obstacle constructed so that it contains a system of transparent and nontransparent circular sectors alternatively spaced, as in Figure 2. The solid angle is divided into $2N$ equal sectors (Urcid and Padila 2005), being nontransparent and transparent alternatively for the incident light. The incident beam has the amplitude with a magnitude decreasing toward the center of the obstacle. Suppose that the center of intersection of all the sectors is a point which is transparent for the incident light. On the screen placed in the distance z from the obstacle, a diffraction pattern is obtained on the plane parallel to the plane of the obstacle.

Complex amplitude of the diffracted light with Gaussian distribution of amplitude is determined starting from the Fresnel-Kirchhoff's diffraction formula for complex amplitude [Born and Wolf 1980],

$$\Psi(P) = -\frac{iA}{2\lambda}\iint_S \frac{e^{ik(r+s)}}{rs}\left[\cos(n,r) - \cos(n,s)\right] dS$$

at a point P on the observation plane, where r and s are the distances from the source to the aperture and from the aperture to an arbitrary point P, respectively. Modifying this formula for complex amplitude of the diffracted light with Gaussian distribution of amplitude, the complex amplitude is determined solving the integral [Moser, Bejtullahu 1981], [Bejtullahu, Janicijevic, Moser and Jonoska 1984],

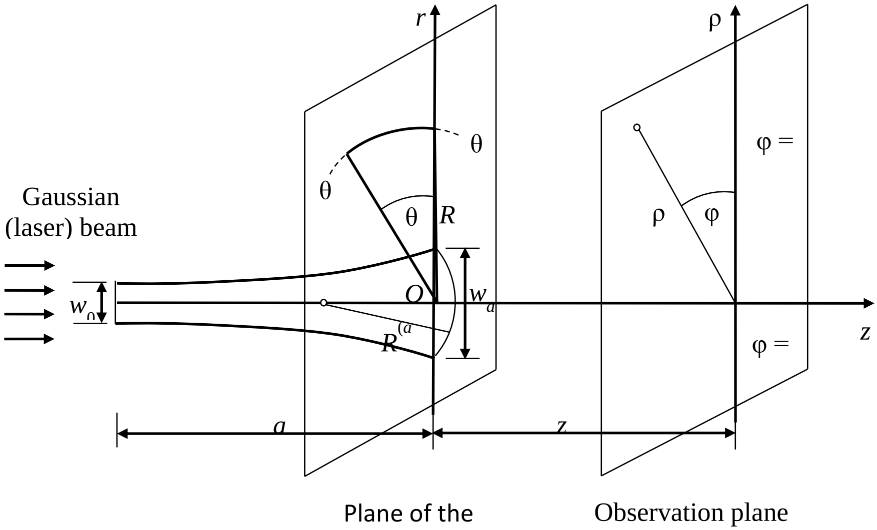

Fig. 2: For determining the complex amplitude of diffraction on a circular sector by a Gaussian beam.

$$\Psi(\rho,\varphi,z)=\frac{ik}{2\pi z}\exp\left[-ik\left(z+\frac{\rho^2}{2z}+\delta\right)\right]\frac{w_0}{w^{(a)}}\iint_S e^{-\frac{r^2}{w^{(a)2}}}e^{-i\left[\frac{k}{2}\left(\frac{1}{R^{(a)}}+\frac{1}{z}\right)\right]r^2+\frac{k}{z}\rho r\cos(\theta-\varphi)}rdrd\theta \qquad (10)$$

In this formula a is the distance from the obstacle to the narrowest cross-section of the Gaussian beam of light, $w^{(a)}=w_0\sqrt{1+\left(\frac{2a}{kw_0}\right)^2}$ is the radius of the Gaussian beam at the plane of the obstacle, $R^{(a)}=a\left[1+\left(\frac{2a}{kw_0}\right)^2\right]$ is the radius of curvature of the phase surfaces of the incident Gaussian beam, w_0 is the minimum radius of the incident Gaussian beam (when $a=0$), $\delta=\frac{1}{k}\arctan\left(\frac{2a}{kw_0}\right)$ is a phase constant depending on the parameters of the incident beam, and k is the wave number. r and θ are the polar coordinates at the plane of the obstacle, and ρ and φ are the polar coordinates at a point of the diffraction plane where the intensity of the diffraction pattern is determined [Ishchenko 1980], [Yariv A. and Pochi Y. 2006] (see Figure 2). Integration must be taken over all parts transparent for the light, at the plane of the obstacle. Integration over the variable r must be taken over 0 to the radius of the obstacle R. Since R can be taken very large in comparison with the width of the Gaussian beam, integration over r can be taken at infinity, because the transparency diminishes with the distance from the beam axis. Integration over the variable θ, π must be divided into N equal parts and then taking the summation of integrals over all the openings.

Integral (10) can be rearranged into the form (Watson 1944):

$$\Psi(\rho,\varphi,z)=K\sum_{m=1}^{2N}(-1)^m\int_0^{\infty}\exp\left[-\left(\frac{1}{w^{(a)2}}+i\frac{k}{2f}\right)r^2\right]rdr\int_0^{\psi_m}e^{(-ibr\cos\gamma)}d\gamma \qquad (11)$$

In the second integral of (11), the definition formula for the unknown cylindrical function [Agrest and Maksimov 1972]:

$$E_0^-(x,\eta)=\frac{2}{\pi}\int_0^{x}e^{(-ib\eta\cos\gamma)}d\gamma \qquad (12)$$

can be used to write instead of (10):

$$\Psi(\rho,\varphi,z)=\frac{\pi}{2}K\sum_{m=1}^{2N}(-1)^m\int_0^{\infty}E_0^-(\psi_m,br)e^{-p^2r^2}rdr \qquad (13)$$

Now we use the relation for incomplete cylindrical functions [Watson 1944]:

$$E_0^-(\psi_m,br)=J_0(\psi_m,br)-iH_0(\psi_m,br) \qquad (14)$$

where $J_0(\psi_m,br)$ and $H_0(\psi_m,br)$ are the Bessel and Struve incomplete functions of zero order. Substituting into (11), it follows:

$$\Psi(\rho,\varphi,z)=\frac{\pi}{2}K\sum_{m=1}^{2N}(-1)^m\int_0^{\infty}\left[J_0(\psi_m,br)-iH_0(\psi_m,br)\right]e^{-p^2r^2}rdr \qquad (15)$$

In relation (15) two integrals appear, called Weber's exponential integrals for incomplete Bessel and Struve functions [Harris and Fripiat 2009]:

$$J_1=\int_0^{\infty}J_0(\psi_m,br)e^{-p^2r^2}rdr \qquad (16)$$

$$J_2 = \int_0^\infty H_0\left(\psi_m, br\right) e^{-p^2 r^2} r dr \tag{17}$$

From (14) and (15), it follows:

$$\Psi\left(\rho, \varphi, z\right) = \frac{\pi}{2} K \sum_{m=1}^{2N} (-1)^m \left(J_1 - iJ_2\right) \tag{18}$$

According to [Jones 2007], we have:

$$J_1 = \frac{1}{\pi}\left[\frac{1}{p^2} - \frac{b}{p^3}\int_0^{\psi_m} \cos\gamma \cdot F\left(\frac{b\cos\gamma}{2p}\right) d\gamma\right] \tag{19}$$

$$J_2 = \frac{1}{2\sqrt{\pi}p^2}\int_0^{\psi_m} \cos\gamma \cdot e^{-b^2 \frac{\cos\gamma}{4p^2}} d\gamma \tag{20}$$

Solution of the integral (20) yields:

$$J_2 = -\frac{i}{2p^2}\exp\left[-\frac{b^2}{4p^2}\right]\mathrm{Erf}\left(\frac{ib\sin\psi_m}{2p}\right) \tag{21}$$

Substituting (19) and (21) into (15), the complex amplitude of the diffracted light is obtained.

4. IRRADIANCE OF THE DIFFRACTION PATTERN

To obtain the irradiance of diffraction pattern, the complex amplitude $\Psi\left(\rho, \varphi, z\right)$ must be multiplied by its complex conjugate. Since all the results are complex, real and imaginary parts of the expression must be separated. Rearranging J_1 and J_2 we can express the complex amplitude $\Psi\left(\rho, \varphi, z\right)$ now separated into the real and imaginary parts, in the form:

$$\Psi\left(\rho, \varphi, z\right) = \frac{K\sqrt{\pi}}{2p^2}(A + iB) \tag{22}$$

where:

$$A = -\sum_{m=1}^{2N} (-1)^m \left\{ \sum_{n=0}^{\infty} \frac{(-1)^n}{a_n} A_{n+1} \left[\sum_{s=0}^{n} b_n \sin\left[2(n+1-s)\psi_m\right] + c_n \psi_m \right] + \right.$$
$$\left. + \frac{\sqrt{\pi}}{2} \exp\left[-\frac{b^2\left(\beta^2-\alpha^2\right)}{4\left(\beta^2+\alpha^2\right)^2} \right] \left[\cos\gamma \operatorname{Re}\operatorname{Erf}(Q) - \sin\gamma \operatorname{Im}\operatorname{Erf}(Q)\right] \right\} \tag{23}$$

$$B = \sum_{m=1}^{2N} (-1)^m \left\{ \sum_{n=0}^{\infty} \frac{(-1)^n}{a_n} A_n \left[\sum_{s=0}^{n} b_n \sin\left[2(n+1-s)\psi_m\right] + c_n \psi_m \right] - \right.$$
$$\left. - \frac{\sqrt{\pi}}{2} \exp\left[-\frac{b^2\left(\beta^2-\alpha^2\right)}{4\left(\beta^2+\alpha^2\right)^2} \right] \left[\sin\gamma \operatorname{Re}\operatorname{Erf}(Q) - \cos\gamma \operatorname{Im}\operatorname{Erf}(Q)\right] \right\} \tag{24}$$

where $Q = \frac{ib}{2p}\sin\varphi$ is the argument of Erf function (Abramowitz and Stegun 1965).

After rearranging expressions (22), (23) and (24) we obtain for the irradiance:

$$\mathscr{I} = \frac{4c^2 f^2 w^{(a)2} w_0^2}{4f^2 + k^2 w^{(a)4}} \left(A^2 + B^2\right) \tag{25}$$

Since the expression for complex amplitude $\Psi(\rho,\varphi,z)$ is very complicated and cumbersome, a mathematical analysis cannot be carried out for this function, as well as for the function of irradiance for apertures of various shapes. Nevertheless, a numerical interpretation can be given only for the Fraunhofer case. In a special case for one sector, we obtain for the irradiance:

$$\mathscr{I} = \frac{4c^2 f^2 w^{(a)2} w_0^2}{4f^2 + k^2 w^{(a)4}} \exp\left[-\frac{b^2}{2w^{(a)2}} \frac{4f^2 w^{(a)4}}{4f^2 + k^2 w^{(a)4}} \right] \left[(1 - \operatorname{Re}\operatorname{Erf} Q)^2 + (\operatorname{Im}\operatorname{Erf} Q)^2 \right] \tag{26}$$

5. DISCUSSION

Formula (26) contains three factors. The first factor, if the incident beam is determined, for an unchanged distance from the diffraction pattern, is a constant. The second factor is characteristic for Gaussian beams and shows that irradiance is decreasing with the distance from the center of the beam. Only the third factor is dependent on the variable φ. If for φ we take the special values 0 and π, then according to (47) $Q = 0$ and this factor equals 1. Analysis of these results indicates the fact that diffraction fringes obtained in this case are closed lines. Using a different mathematical approach, it can be shown that these lines are ellipses.

6. CONCLUSION

We have shown that making use of incomplete Bessel, Hanckel, and Struve cylindrical functions of the Poisson integral representation form, leading to Weber integrals, we can obtain the complex amplitude of the diffracted Gaussian beam by circular sectors, which is in accordance with previously obtained results by other methods (Urcid and Padila 2005). This approach can be beneficial to the study of the diffraction phenomena, in particular in the case of the Gaussian beam diffraction. The rigorous mathematical solution using incomplete Weber integrals allows to reveal the new qualitative characteristic of the diffraction on apertures of more complicated shapes. Formula (26), containing the three factors discussed, illustrates the possibility of this approach to describe diffraction phenomena in the most correct and instructive manner.

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