# REMARKS ON INVERSE Γ-SEMIGROUPS

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## ABSTRACT

Lately, several kinds of inverse Γ-semigroups have been defined, some of which appear to have properties similar to those of inverse semigroups, while the other types lack those sorts of results that would justify their study. The present paper aims to prove the non-existence of those types of inverse Γ-semigroups by relating them with a semigroup which can be always associated with any given Γ-semigroup.
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## 1. INTRODUCTION AND PRELIMINARIES

If *S* and Γ are two nonempty sets, then every map : $S \times \Gamma \times S \to S$ will be called a Γ-multiplication in *S*. The result of this multiplication for $a, b \in S$ and $\gamma \in \Gamma$ is denoted by $a\gamma b$. Sen and Saha (1986) said that a Γ-semigroup *S* is an ordered pair $(S, (\cdot)_\Gamma)$ equipped with a Γ-multiplication on *S* which satisfies the following property:

$$\forall (a, b, c, \alpha, \beta) \in S \times \Gamma, (a\alpha b)\beta c = a\alpha(b\beta c).$$

An element *a* of a Γ-semigroup $(S, (\cdot)_\Gamma)$ is called regular if there is some $b \in S$ and $(\alpha, \beta) \in \Gamma \times \Gamma$ such that $a = a\alpha b\beta a$. We observe here that the element $b\beta a\alpha b$ satisfies two equations:

$$a = a\alpha(b\beta a\alpha b)\beta a \text{ and}$$
$$b\beta a\alpha b = (b\beta a\alpha b)\beta a\alpha(b\beta a\alpha b).$$

This situation motivates the following definition. Given Γ-semigroup $(S,(\cdot)_\Gamma)$ and $a \in S$, we say that $b \in S$ is an $(\alpha,\beta)$ −inverse of *a* where $(\alpha,\beta) \in \Gamma \times \Gamma$ if $a = a\alpha b\beta a$ and $b = b\beta a\alpha b$. The set of $(\alpha,\beta)$ −inverses of *a* is denoted by $V_\alpha^\beta(a)$. Back to the definition of regular elements, we observe that if an element is regular, then it has a regular inverse. It is expected in similarity with regular semigroups that an element may have several inverses. This situation is a bit messy especially with Γ-semigroups. Consequently, Saha and Seth (1987-1988) defined inverse Γ-semigroups as those regular Γ-semigroups $(S,(\cdot)_\Gamma)$ with the property that for every $a \in S$, $\left|V_\alpha^\beta(a)\right| = 1$ whenever there is an $(\alpha,\beta)$ −inverse of *a*. In other words, every element of an inverse Γ-semigroup has a unique $(\alpha,\beta)$ −inverse for some pair $(\alpha,\beta) \in \Gamma \times \Gamma$. Such Γ-semigroups are renamed as inverse Γ-semigroups of the first kind in (Beqiri and Petro 2015 a:b; Beqiri 2017). Another kind of inverse Γ-semigroups defined in (Beqiri 2017) are those called there inverse Γ-semigroups of the second kind which by definition are those Γ-semigroups satisfying the property that for every $a \in S$ there exists a unique $(\alpha,\beta) \in \Gamma \times \Gamma$ and a unique $b \in V_\alpha^\beta(a)$. It is tempting to consider those inverse Γ-semigroups of the second kind with the property that the only pair of $\Gamma \times \Gamma$ for which *a* has an inverse is the pair $(\alpha,\beta)$. We call such Γ-semigroups strong inverse Γ-semigroups of the second kind.

It seems at a first look that strong inverse Γ-semigroups of the second kind are closer to inverse semigroups than the other types and therefore more promising for future study. In fact, this impression turns out to be wrong. The first indication for this was given in (Beqiri 2017) where the study of the broader class of the inverse Γ-semigroups of the second kind was avoided with the argument that there are no available non-trivial examples of such Γ-semigroups that would motivate their study.

In this present note we prove that strong inverse Γ-semigroups can never exist and to achieve this we relate any given Γ-semigroup $(S,(\cdot)_\Gamma)$ to a certain semigroup $\Omega_{\gamma_0}$ for which we prove that it is an inverse semigroup whenever $(S,(\cdot)_\Gamma)$ is a strong inverse Γ-semigroup of the second kind. But it can be proved very easily that $\Omega_{\gamma_0}$ can never be an inverse semigroup, so in return we obtain that strong inverse Γ-semigroups of the second kind do not really exist. Also, we prove that inverse Γ-semigroup of the first kind do not exist either, if the Γ-semigroup satisfies a certain disconnectedness condition which says roughly that different elements of S have disjoint set of ”operators” from Γ for which they have inverses. These two results show how risky it is to consider in the theory of Γ-semigroups axiomatic systems analogous with those of semigroup theory which determine inverse semigroups there.

## 2. MAIN RESULTS

The semigroup $\Omega_{\gamma_0}$ is defined for the first time in (Çullaj and Krakulli 2020) and is similar to the semigroup $\Sigma_{\gamma_0}$defined in (Pasku 2017), but in contrast with (Pasku 2017) where the elements of Γ, regarded as elements of $\Sigma_{\gamma_0}$, formed there a left zero semigroup, in (Çullaj and Krakulli 2020) they form a group which allows us to tackle with problems related with the regularity of the Γ-semigroup. Based on (Çullaj and Krakulli 2020), we give here briefly the construction of $\Omega_{\gamma_0}$ and mention some elementary facts about it. The definition of $\Omega_{\gamma_0}$ uses the fact that we can always define a multiplication • on any nonempty set Γ in such a way that $(\Gamma,\bullet)$ becomes a group. This in fact is equivalent to the axiom of choice (Hajnal and Kertsz 1972). Further, let $(F,\ \cdot)$ be the free semigroup on $S$. Its elements are finite strings $(x_1,\ldots,x_n)$ where each $x_i \in S$ and the product is the concatenation of words. Now we define $\Omega_{\gamma_0}$ as the quotient semigroup of the free product $F * \Gamma$ of $(F,\ \cdot)$ with $(\Gamma,\ \bullet)$ by the congruence generated from the set of relations

$$((x,y),x\gamma_0 y),((x,\gamma,y),x\gamma y)$$

for all $x,y \in S, \gamma \in \Gamma$ and with $\gamma_0 \in \Gamma$ a fixed element. We can also regard the group $(\Gamma,\bullet)$ as given by a presentation with generators the elements of Γ, and relations arising from the multiplication table of the group. So, a presentation of $\Omega_{\gamma_0}$ has now as a generating set $S \cup \Gamma$, and relations those mentioned above together with those arising from the multiplication table of $(\Gamma,\ \bullet)$. The following is Lemma 2.1 of (Cullaj and Krakulli 2020) which we give here without a proof.

**Lemma 2.1** *Every element of* $\Omega_{\gamma_0}$ *can be represented by an irreducible word which has the form* $(\gamma,x,\gamma'),(\gamma,x),(x,\gamma),\gamma$ *or* $x$ *where* $x \in S$ *and* $\gamma,\gamma' \in \Gamma$.

In what follows, we prove that strong inverse Γ-semigroups of the second kind do not exist at all. The proof uses a relationship that exists between $(S,(\cdot)_\Gamma)$ and its associate $\Omega_{\gamma_0}$. The following theorem reveals this relationship.

**Proposition 2.1** $(S,(\cdot)_\Gamma)$ *is a strong inverse Γ-semigroups of the second kind if and only if* $\Omega_{\gamma_0}$ *is an inverse semigroup.*

**Proof.** Assume that *S* is a strong inverse Γ-semigroup of the second kind. This means that for every $a \in S$ there is a unique, $(\gamma_1,\gamma_2) \in \Gamma \times \Gamma$ and for which it exists a unique $x \in S$ such that $a = a\gamma_1 x \gamma_2 a$. It is straightforward that *a* has a $(\gamma_1,\gamma_2)$- inverse in $(S,(\cdot)_\Gamma)$ which is $x\gamma_2 a\gamma_1 x$, and as $(S,(\cdot)_\Gamma)$ is an inverse Γ-semigroup of the second kind, we have that $x\gamma_2 a\gamma_1 x = x$. Observe now that in $\Omega_{\gamma_0}$ the element *a* has an inverse which is $(\gamma_1 x \gamma_2)$. This is true since

$$a = a\gamma_1 x \gamma_2 a \text{ and}$$
$$(\gamma_1 x \gamma_2)a(\gamma_1 x \gamma_2) = \gamma_1 x \gamma_2.$$

This inverse is in fact unique, for if $\alpha_1 y \alpha_2 \in \Omega_{\gamma_0}$ was another inverse, where $\alpha_1$ or $\alpha_2$ can be possibly empty operators, then

$$a = a\alpha_1 y \alpha_2 a \text{ and } (\alpha_1 y \alpha_2)a(\alpha_1 y \alpha_2) = \alpha_1 y \alpha_2.$$

The second equality implies that $y = y\alpha_2 a\alpha_1 y$, and the first implies that $y\alpha_2 a\alpha_1 y$ is an $(\alpha_1,\alpha_2)$-inverse of *a* in *S*, hence from the assumption on *S* we have that $\alpha_1 = \gamma_1$, $\alpha_2 = \gamma_2$ and $y\alpha_2 a\alpha_1 y = x$

This last equality implies that $x = y$, and as a consequence we have that $\alpha_1 y \alpha_2 = \gamma_1 x \gamma_2$. We show that the same happens with all the remaining types of elements of $\Omega_{\gamma_0}$. Let $\alpha_1 a \alpha_2$ be another type of element of $\Omega_{\gamma_0}$. An inverse in $\Omega_{\gamma_0}$ is the element $\alpha_2^{-1}\gamma_1 x \gamma_2 \alpha_1^{-1} \in \Omega_{\gamma_0}$ , since

$$(\alpha_1 a \alpha_2)(\alpha_2^{-1}\gamma_1 x \gamma_2 \alpha_1^{-1})(\alpha_1 a \alpha_2) = \alpha_1 a \gamma_1 x \gamma_2 a \alpha_2 = \alpha_1 a \alpha_2,$$

and

$$(\alpha_2^{-1}\gamma_1 x \gamma_2 \alpha_1^{-1})(\alpha_1 a \alpha_2)(\alpha_2^{-1}\gamma_1 x \gamma_2 \alpha_1^{-1}) = \alpha_2^{-1}\gamma_1(x\gamma_2 a \gamma_1 x)\gamma_2 \alpha_1^{-1}$$
$$= \alpha_2^{-1}\gamma_1 x \gamma_2 \alpha_1^{-1}$$

This inverse is unique for if $\beta_1 y \beta_2$ was another inverse, then in $\Omega_{\gamma_0}$ we would have

$$\alpha_1 a \alpha_2 = (\alpha_1 a \alpha_2)(\beta_1 y \beta_2)(\alpha_1 a \alpha_2)$$

$$= \alpha_1(a(\alpha_2\beta_1)y(\beta_2\alpha_1)a)\alpha_2,$$

which implies that $a = a(\alpha_2\beta_1)y(\beta_2\alpha_1)a$. But we would also have that

$$(\beta_1 y \beta_2)(\alpha_1 a \alpha_2)(\beta_1 y \beta_2) = \beta_1 y \beta_2,$$

which implies that $y(\beta_2\alpha_1)a(\alpha_2\beta_1)y = y$. The assumption on $S$ implies that $y = x$, and $\alpha_2\beta_1 = \gamma_1$, $\beta_2\alpha_1 = \gamma_2$, or equivalently, $\beta_1 = \alpha_2^{-1}\gamma_1$ and $\beta_2 = \gamma_2\alpha_1^{-1}$. As a result, we have that $\beta_1 y \beta_2 = \alpha_2^{-1}\gamma_1 x \gamma_2\alpha_1^{-1}$ which proves uniqueness. Further we see that also $\alpha a \in \Omega_{\gamma_0}$ has an inverse which is $\gamma_1 x \gamma_2 \alpha^{-1} \in \Omega_{\gamma_0}$, because

$$\begin{aligned}(\alpha a)(\gamma_1 x \gamma_2 \alpha^{-1})(\alpha a) &= \alpha a \gamma_1 x \gamma_2 a \\ &= \alpha a,\end{aligned}$$

and

$$(\gamma_1 x \gamma_2 \alpha^{-1})(\alpha a)(\gamma_1 x \gamma_2 \alpha^{-1}) = \gamma_1 (x \gamma_2 a \gamma_1 x) \gamma_2 \alpha^{-1} = \gamma_1 x \gamma_2 \alpha^{-1}.$$

This inverse is unique for if $\beta_1 y \beta_2$ was another inverse, then in $\Omega_{\gamma_0}$ we would have on the one hand that

$$\begin{aligned}\alpha a &= (\alpha a)(\beta_1 y \beta_2)(\alpha a) \\ &= \alpha(a \beta_1 y (\beta_2 \alpha) a),\end{aligned}$$

which implies that $a = a\beta_1 y(\beta_2\alpha)a$ and on the other hand that

$$(\beta_1 y \beta_2)(\alpha a)(\beta_1 y \beta_2) = \beta_1 y \beta_2,$$

which implies that $y = y(\beta_2\alpha)a\beta_1 y$. The assumption on $S$, implies that $\beta_1 = \gamma_1$, $\beta_2 = \gamma_2\alpha^{-1}$ , and $y = x$, therefore uniqueness. A similar proof is available for elements of the form $a\alpha \in \Omega_{\gamma_0}$therefore we have omitted it. Finally, every $\alpha \in \Gamma$ has a unique inverse $\alpha^{-1}$, which is its inverse in $(\Gamma, \bullet)$. This can be seen easily by discarding from the list of possible inverses of $\alpha$ all the elements of $S \cup \Gamma S \cup S\Gamma \cup \Gamma S \Gamma$.

For the converse, if $\Omega_{\gamma_0}$ is an inverse semigroup, then every $a \in S$ has an inverse in $\Omega_{\gamma_0}$. We will show that every $a \in (S, ()_\Gamma)$ has a unique inverse in $(S, ()_\Gamma)$. For this we distinguish between the following four cases. First, if the inverse of $a$ in $\Omega_{\gamma_0}$ is of the form $\alpha x \beta$ where $x \in S$, then $a\alpha x \beta a = a$ and $\alpha x \beta = (\alpha x \beta)a(\alpha x \beta)$, hence $x = x\beta a \alpha x$. Both equalities mean that $x$ is an $(\alpha, \beta)$-inverse of $a$ in $(S, ()_\Gamma)$. This is unique, since in contrary, if there were $\gamma, \delta \in \Gamma$ and

$y \in S$ such that $a = a\gamma y\delta a$ and $y = y\delta a\gamma y$, then in $\Omega_{\gamma_0}$, $(\gamma y\delta)a(\gamma y\delta)$ is an inverse of *a*, but $\Omega_{\gamma_0}$ is an inverse semigroup, so

$$
\begin{aligned}
\alpha x\beta &= (\gamma y\delta)a(\gamma y\delta) \\
&= \gamma(y\delta a\gamma y)\delta \\
&= \gamma y\delta,
\end{aligned}
$$

and then $\gamma = \alpha, y = x$ and $\delta = \beta$. Second, if $\alpha x$ is the inverse of *a* in $\Omega_{\gamma_0}$, then $a(\alpha x)a = a$ and $\alpha xa\alpha x = \alpha x$. We can rewrite these as $a\alpha x\gamma_0 a = a$ and $\alpha x\gamma_0 a\alpha x = \alpha x$. The second equation is equivalent to $x\gamma_0 a\alpha x = x$. In terms of $(S, (\cdot)_\Gamma)$ these equalities mean that *x* is a $(\alpha, \gamma_0)$-inverse of *a*. Similarly to the first case, if there are $\gamma, \delta \in \Gamma$ and *y* $\in S$ such that $a = a\gamma y\delta a$ and $y = y\delta a\gamma y$, then $\gamma y\delta = \alpha x\gamma_0$, and then $\gamma = \alpha, y = x$ and $\delta = \gamma_0$. Third, the inverse of *a* in $\Omega_{\gamma_0}$ is some $x\alpha$. This case is dealt with similarly to the second case. Fourth, the inverse of *a* in $\Omega_{\gamma_0}$ is some $x \in S$. Then, $axa = a$ and $x = xax$, or equivalently, $a\gamma_0 x\gamma_0 a = a$ and $x = x\gamma_0 a\gamma_0 x$, which imply that *x* in a $(\gamma_0, \gamma_0)$-inverse of *a* in $(S, (\cdot)_\Gamma)$. To prove uniqueness we assume that there are $\gamma, \delta \in \Gamma$ and $y \in S$ such that $a = a\gamma y\delta a$ and $y = y\delta a\gamma y$. Then, the same as before, we have $\gamma y\delta = \gamma_0 x\gamma_0$, and $\gamma = \gamma_0 = \delta, y = x$. ∎

**The nonexistence of strong inverse Γ-semigroups of the second kind.**

**Theorem 2.1** *There are no strong inverse* Γ*-semigroups of the second kind.*

**Proof.** Assume that $(S, (\cdot)_\Gamma)$ is a strong inverse Γ-semigroups of the second kind, then from proposition **2.1** the semigroup $\Omega_{\gamma_0}$ is an inverse semigroup. Let *a*, *b* $\in S$ arbitrary elements, and let $x \in S$ be an $(\gamma_1, \gamma_2)-$inverse of *a* in $(S, (\cdot)_\Gamma)$, and also let $y \in S$ be an $(\beta_1, \beta_2)-$ inverse of *b* in $(S, (\cdot)_\Gamma)$. It follows that $\gamma_1 x\gamma_2$ is the inverse of *a* in $\Omega_{\gamma_0}$, and $\beta_1 x\beta_2$ is the inverse of *b* in $\Omega_{\gamma_0}$. Since $\Omega_{\gamma_0}$ is an inverse semigroup, then the idempotents $a\gamma_1 x\gamma_2$ and $\beta_1 y\beta_2 b$ commute in $\Omega_{\gamma_0}$. Therefore:

$$
\begin{aligned}
a\gamma_1 x(\gamma_2\beta_1)y\beta_2 b &= (a\gamma_1 x\gamma_2)(\beta_1 y\beta_2 b) \\
&= (\beta_1 y\beta_2 b)(a\gamma_1 x\gamma_2) \\
&= \beta_1 y\beta_2(b\gamma_0 a)\gamma_1 x\gamma_2.
\end{aligned}
$$

So, we have that $a\gamma_1 x(\gamma_2\beta_1)y\beta_2 b = \beta_1 y\beta_2(b\gamma_0 a)\gamma_1 x\gamma_2$ which is an impossible equality in $\Omega_{\gamma_0}$. This contradiction proves the nonexistence of a strong inverse $(S,(\cdot)_\Gamma)$ of the second kind.■

**The nonexistence of inverse Γ-semigroups of the first kind in a special case.**

Regarding the inverse Γ-semigroups of the first kind we prove that, under certain circumstances which we will describe below, they do not exist either. Let $(S,(\cdot)_\Gamma)$ be an inverse Γ semigroup of the first kind. Assume that Γ is partitioned as a nontrivial disjoint union of subsets $\Gamma = \bigsqcup_{\alpha\in S}\Gamma_\alpha$ where each $\Gamma_\alpha$ has the property that for every pair $(\alpha,\beta)\in\Gamma\times\Gamma$ for which an $(\alpha,\beta)$ −inverse of *a* exist, then this inverse is unique, and both $\alpha,\beta\in\Gamma_\alpha$. This nontrivial partition of Γ would not be possible for inverse Γ-semigroups of the third kind since in that case there is only one $\Gamma_\alpha$, namely Γ. We call disconnected inverse Γ-semigroups of the first kind every inverse Γ-semigroups of the first kind which satisfy the above condition. In fact, the following proposition shows that such Γ-semigroups do not really exist.

**Theorem 2.2** *There are no disconnected inverse* Γ*-semigroups of the first kind.*

**Proof.** Assume that $(S,(\cdot)_\Gamma)$ is a disconnected inverse Γ-semigroups of the first kind. We will define e new $\Gamma'$-semigroup $(S,(\cdot)_{\Gamma'})$ where $\Gamma'\subseteq\Gamma$ and that $(S,(\cdot)_{\Gamma'})$ is a strong inverse $\Gamma'$- semigroup of the second kind. This contradiction will prove the nonexistence in our case. To define $\Gamma'$ we will proceed as follows. For every $a\in S$ we chose some $(\alpha,\beta)\in\Gamma\times\Gamma$ for which an $(\alpha,\beta)$-inverse of $a$ exists. If it happens that *a* has no $(\beta,\alpha)$-inverse in *S*, then we define $\Gamma'_a = \{\alpha,\beta\}$. Otherwise, if there is a $(\beta,\alpha)$-inverse of *a*, then we prove first that there is also an $(\alpha,\alpha)$-inverse of *a* in *S*. Indeed, if *x*, *y* $\in S$ are $(\alpha,\beta)$ and $(\beta,\alpha)$-inverses of *a* respectively, then

$$a = a\alpha x\beta a \text{ and } a = a\beta y\alpha a,$$

from which we get that

$$a = a\alpha(x\beta a\beta y)\alpha a$$

and

$$\begin{aligned}(x\beta a\beta y)\alpha a\alpha(x\beta a\beta y) &= x\beta(a\beta y\alpha a)\alpha(x\beta a\beta y)\\ &= x\beta a\alpha(x\beta a\beta y)\\ &= x\beta(a\alpha x\beta a)\beta y\\ &= x\beta a\beta y,\end{aligned}$$

proving that $a$ has an $(\alpha, \alpha)$-inverse, as claimed. In this case, we define $\Gamma'_a = \{\alpha\}$. We do this for every $a \in S$ to obtain a subset $\Gamma'_a$ of $\Gamma_a$ and then define $\Gamma' = \sqcup_{a\in S} \Gamma'_a$. Now there is an obvious $\Gamma'$-semigroup $(S, (\cdot)_{\Gamma'})$ where the $\Gamma'$-multiplication $(\cdot)_{\Gamma'}$ is the one induced by the restriction in $\Gamma'$. Finally, in $(S, (\cdot)_{\Gamma'})$ we have that for every element $a \in S$, there is a unique $(\alpha, \beta) \in \Gamma' \times \Gamma'$ for which a unique $x \in S$ exists such that $x$ is a $(\alpha, \beta)$-inverse of $a$. The uniqueness of $x$ follows from the fact that $(S, (\cdot)_{\Gamma})$ is adisconnected inverse Γ-semigroup of the first kind, and the uniqueness of $(\alpha, \beta)$ follows from the disconnectedness of $(S, (\cdot)_{\Gamma})$ together with the fact that $\Gamma'_a$ is the only component of the partition of $\Gamma'$ which contains a unique $\alpha$ and a unique $\beta$ such that $x$ is an $(\alpha, \beta)$-inverse of $a$. Summarizing, $(S, (\cdot)_{\Gamma'})$ is a strong inverse $\Gamma'$-semigroup of the second kind and we are done.■

**REFERENCES**

**Beqiri O. 2017.** Γ-*gjysmëgrupet inversivë dhe* Γ-*unazat inversive*, Tezë doktorature, Uni- versiteti i Tiranës,

**Beqiri O, Petro P. 2015 a.** *Weak completely inverse* Γ-*semigroups*. *Bulletin of the Calcutta Mathematical Society*. **107 (5):** 421428.

**Beqiri O. Petro P. 2015 b.** *Weak completely inverse* Γ-*semigroups*. *Bulletin of the Calcutta Mathematical Society*. **107 (2):** 117-124

**Çullhaj F, Krakulli A. 2020** Some characterizations of regularity and intra-regularity of Γ-semigroups by means of quasi-ideals. Accepted for publication in *Novi Sad Journal of Mathematics*. https://doi.org/10.30755/NSJOM.10795.

**Hajnal A, Kertsz A. 1972**. Some new algebraic equivalents of the axiom of choice. *Publicationes Mathematicae Debrecen.* **19:** 339-340.

**Howie JM. 1995.** Fundamentals of semigroup theory. Clarendon Press Oxford, First Edition.

**Pasku E. 2017.** *The adjoint semigroup of a* Γ *semigroup*. *Novi Sad Journal of Mathematics*. **47 (2):** 31-39.

**Saha NK. 1987**. On Γ-semigroup. *Bulletin of the Calcutta Mathematical Society*. **79:** 331-335.

**Sen MK, Saha NK. 1986.** On Γ-semigroup I. *Bulletin of the Calcutta Mathematical Society*.**78:** 180-186.

**Saha NK, Seth A. 1987-1988.** Inverse Γ-semigroup. *Journal of Pure Math*. **6**: 0-8.